Can the cosmological "constant" run? - It may run

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ABSTRACT

Using standard quantum field theory, we discuss several theoretical aspects of the possible running of the cosmological constant (CC) term in Einstein's equations. The basic motivation for the present work is to emphasize that this possibility should also be taken into account when considering dynamical models for the dark energy (DE), which are nowadays mainly focused on identifying the DE with the energy density associated to one or more *ad hoc* scalar fields. At the same time, we address some recent criticisms that have been published (or privately communicated to us) attempting to cast doubts on the fundamental possibility of such running. In this work, we argue that while there is no comprehensive proof of the CC running, there is no rigorous proof of the non-running either. In particular, some purported "non-running theorem" recently adduced in the literature is, in our opinion, completely insubstantial and formally incorrect. The way to the CC running is, therefore, still open and we take here the opportunity to present a pedagogical review of the present state of the art in this field, including a brief historical account.

1   Introduction

The relevance of the cosmological constant (CC) problem (in its various aspects [1]) has triggered a vast interest in identifying the possible quantum effects on the vacuum energy density and their potential implications in cosmology. Many of the proposed approaches to the CC problem are based on considering that the vacuum energy is related, or even superseded, by the specific properties of some ad hoc scalar field (or collection of them) [2]. This point of view is what has finally led to the general notion of dark energy (DE) in its manifold possible

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forms [3]. However, in this work, we will essentially concentrate on the cosmological constant term in Einstein's equations and, more generally, on the quantum field theoretical aspects of the renormalizable vacuum action that ensues after appropriately extending the original Einstein-Hilbert action. This, more fundamental, QFT point of view naturally leads to consider the renormalization group (RG) properties of the parameters of the renormalizable vacuum action, in particular of the CC term, Λ, in it. One expects that the physical CC resulting from this RG approach can be viewed as a running entity that could perhaps explain, in truly fundamental terms, the suspected dynamical nature of the DE, which has long been invoked in order to explain other intriguing aspects of the CC conundrum, such as the cosmic coincidence problem [2, 3].

There are already many papers on the subject under consideration, and one can distinguish different approaches. First of all, there is the great CC problem or "old CC problem" (approach A), i.e. the formidable task of trying to explain the relatively small (for Particle Physics standards) measured value of the CC [4], roughly $\rho_\Lambda^{\exp} \sim 10^{-47} \, GeV^4$, after the many phase transitions that our Universe has undergone since the very early times [1]. Second, we have the approach to the CC problem relying on the renormalization group method (approach B), namely by exploring the possibility that a moderate running of the CC could leave a physical dynamical imprint that can eventually be observed [5]. Notice that this RG viewpoint could also have some bearing on the aforementioned cosmic coincidence problem, see [6]. Let us remark, however, that approach B has no obvious impact on solving the main problem addressed in approach A.

On the other hand, there is a dramatic distinction between the "RG-cosmologies" based on the renormalization group properties of the unknown fundamental physics such as quantum gravity or string theory (sub-approach $\mathbf{I}$) and the alternative renormalization group approach based on the familiar methods and results of the Standard Model (SM) of Particle Physics (sub-approach $\mathbf{II}$) and its favorite extensions (such as e.g. the Minimal Supersymmetric Standard Model [7]). While the implications of the RG method in sub-approach $\mathbf{I}$ are difficult to test in practice, in the case $\mathbf{II}$ the RG technique relies on the well-tested facts of the SM and/or of its extensions which, by the way, will be soon put to the test in the upcoming generation of TeV-class colliders, headed by the imminent startup of the Large Hadron Collider at CERN.

It is pretty obvious that the philosophy of the various combined approaches, e.g., AI and BII, is quite different. In the first case, we meet an attempt to solve the old CC problem using the assumed, but yet unknown, fundamental theory of everything (TOE). The works in this direction deserve a great respect, even though the creation of a successful and falsifiable fundamental TOE looks a rather remote perspective for the time being, exactly as the solution of the great CC problem [1]. In the second case (BII), the purpose is more modest and, perhaps for this reason, more viable; namely, it consists to explore whether the observable CC (whose value is not attempted to be accounted for) can be mildly variable due to the quantum effects generated from the well established quantum field theory (QFT) of Particle Physics, such as the SM and extensions thereof, but in the presence of an external (and dynamical) metric background. The exploration of this possibility, even if looks less ambitious, it represents nevertheless a necessary step on the way to find a fundamental physical explanation for the vacuum energy problems. In this article, we are going to discuss, in a bit more detail than ever before, the quantum field theory aspects of a possible renormalization group running of the vacuum energy.

The paper is organized as follows. In section 2 we present, in a pedagogical way, the preliminaries of the CC running, that is to say, we explain what is the cosmological constant and what

means the renormalization group running. In section 3, we present a brief (and probably incomplete) history of the RG for the vacuum action parameters, and especially for the cosmological constant. We shall provide classification (e.g. *AI* or *BII*, etc) to each of the approaches.

Despite that many of the considerations presented here are rather general, from section 4 on we will concentrate on the approach *BII* and discuss the possibility of having moderate CC running owing to the quantum effects of massive matter fields (e.g., SM constituents). This approach has been first started in [8, 9] and further developed in recent years [10, 11, 12, 13, 14, 15, 16, 17]. Our aim here is to clarify its theoretical background in a consistent form. It is worthwhile to say, from the very beginning, that we do not have a comprehensive proof that the scale dependence (running) of the CC really takes place, but at the same time we emphasize that there is no correct proof of the opposite. Therefore, from our point of view, the relevance of the subject makes perfectly reasonable any sound investigation in this area whether performed within phenomenological or theoretical terms.

Notwithstanding, it is also very important to distinguish these sound investigations from conceptually wrong results in the literature (of which we will comment in detail later on) and thus maintain an unbiased approach to the possibility of CC running. For example, in section 4, we explain the important relationship of the effective potential and its RG invariance with the cosmological constant. It suffices to say here that the "non-running" claim adduced in a recent publication [19] on the sole basis of the RG-invariance of the effective potential, is incorrect and entirely misleading from the conceptual and technical point of view. We explain the mistakes on which that work is based, specially due to the misuse of the RG approach in QFT. Moreover, the technical simplicity of this section enables us to provide a kind of general introduction to the subject, which will hopefully be interesting to the readers.

In section 5 we address other (more serious) arguments against the variable CC, including the ones presented recently by A. Vilenkin [20] and A.D. Dolgov [21]. We hope to convince the reader that, despite these and other arguments (including our own) challenging the possibility of a variable CC, there is still an open window that can accommodate the restricted set of quantum corrections which can provide the physical support for such a running. Finally, in the last section we draw our conclusions.

2   Background notions

Before starting to discuss the possibility and meaning of having CC running, and ultimately of having "RG Cosmology", it is worthwhile to remember what is the definition of the CC, what is the energy of vacuum and what is the renormalization group running in QFT.

2.1   What is Λ

The modern theory of gravitation starts from a classical action that goes beyond the conventional Einstein-Hilbert (EH) action. It includes a more complete structure for the vacuum part, together with the matter:

$$S_{total} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \, (R + 2\Lambda) + S_{HD} + S_{matter} \,. \tag{1}$$

Here, the first term on the *r.h.s.* is the EH action with the cosmological constant Λ; the second term (S_{HD}) includes the higher derivatives that are necessary for the consistency properties (such as renormalizability) of the quantum theory in curved space (see, e.g., [22], [23] or [24] for an introduction) and the last term represents the action of matter, which is responsible for the energy-momentum tensor $T_{\mu\nu}^M = -2(-g)^{-1/2}\delta S_{matter}/\delta g^{\mu\nu}$. It is also customary to include the contribution of the higher derivative term S_{HD} into the energy-momentum tensor; hence the full energy-momentum tensor associated to the action (1) reads $T_{\mu\nu} = -2(-g)^{-1/2}\delta\left(S_{HD} + S_{matter}\right)/\delta g^{\mu\nu}$. Thus, in the absence of matter, we have the vacuum value $T_{\mu\nu}^{vac} = -2(-g)^{-1/2}\delta S_{HD}/\delta g^{\mu\nu}$. This custom gave origin, for instance, to the expression "trace anomaly", given by $\langle T_\mu^\mu \rangle$. This is perhaps an abuse of language, but we shall follow the tradition and use the same wording.

Since our main purpose here is the description of the low-energy gravitational physics, we shall at the moment disregard the higher derivative piece S_{HD} until we start to discuss the quantum effects in curved space-time (see section 5). Therefore, the low energy gravitational physics is well described by just the Einstein dynamical equations for the metric:

$$R_\mu^\nu - \frac{1}{2}R\,\delta_\mu^\nu = 8\pi G\,T_\mu^\nu + \Lambda\,\delta_\mu^\nu\,, \tag{2}$$

where $T_{\mu\nu}$ here is taken to be $T_{\mu\nu}^M$. Let us now consider a cosmological system of matter representing our Universe. According to the Cosmological Principle, on average it is described by an isotropic and homogeneous fluid, with energy-momentum tensor $T_\mu^\nu = -p\,\delta_\mu^\nu + (\rho+p)U^\nu U_\mu$. Therefore, in the locally co-moving frame, where the 4-velocity of the fluid is $U^\mu = (1,0,0,0)$, it takes the simplest form

$$T_\mu^\nu = \text{diag}\left(\rho,\,-p,\,-p,\,-p\right)\,, \tag{3}$$

where ρ and p are the proper density and pressure of matter. It is easy to see that the Λ term on the *r.h.s.* of (2) can be written $\Lambda\,\delta_\mu^\nu = 8\,\pi\,G\,(T_\Lambda)_\mu^\nu$, where the new energy-momentum tensor $(T_\Lambda)_\mu^\nu$ takes the form (3), with the "vacuum energy density" ρ_Λ and "vacuum pressure" p_Λ related as

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} = -p_\Lambda\,. \tag{4}$$

In other words, we have $(T_\Lambda)_\mu^\nu = \rho_\Lambda\,\delta_\mu^\nu$, which justifies ρ_Λ being called the "vacuum energy", or at least a contribution to it. So we shall also use this terminology in what follows.

In order to better understand the role of the curved space-time background for the relevance of the CC term, let us consider the following example. Suppose we take another parametrization (Weyl scaling) for the metric

$$g_{\mu\nu} = \frac{\chi^2}{M^2}\,\bar{g}_{\mu\nu}\,, \tag{5}$$

where $\bar{g}_{\mu\nu}$ is some fiducial metric with fixed determinant. For instance, it can be a flat metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. Furthermore, $\chi = \chi(x^\mu)$ is a scalar field which can be identified as a conformal factor of the metric, M is some mass (perhaps near the Planck mass, M_P). It is easy to see that the CC term looks rather different in these new variables

$$S_\Lambda = -\int d^4x\sqrt{-g}\,\frac{\Lambda}{8\pi G} = -\int d^4x\sqrt{-\bar{g}}\,f\chi^4\,, \qquad \text{where} \qquad f = \frac{\Lambda}{8\pi G\,M^4}\,. \tag{6}$$

The last action is nothing else than the usual potential term for the scalar interaction in χ^4 theory. Equivalently: the inverse image of the χ^4 interaction by the Weyl transformation is an effective CC term in the Einstein frame. One can note that, under the same change of variables, the Einstein-Hilbert term transforms into the action of the scalar field χ with the negative kinetic term and non-minimal conformal coupling [28, 29, 30, 31]. Furthermore, the massive term in the spinor Lagrangian becomes a Yukawa-type interaction between the fermion and the scalar degree of freedom of the metric χ. If we are interested, e.g., in the renormalization of the gravitational action due to the quantum effects of the spinor field, the procedure is completely similar to the well known Nambu-Jona-Lasinio (NJL) model. In this model, one does not quantize the scalar field, exactly as we do not intend to quantize the metric in the approach *BII*. Let us notice in passing that, within the Minimal Subtraction (MS) scheme of renormalization, the RG running of the cosmological constant in the theory of the spinor field is mathematically equivalent to the one of the parameter f in the NJL model [32]. A similar equivalence can be easily established for the quantum effects of a free massive scalar field ϕ – in this case the quantum field interacts with χ via the $\phi^2\chi^2$-term.

2.2   Cosmologies based on a "running" Λ

Next, let us concentrate on the CC term and start by making an important remark, which is at the heart of the message that this paper wishes to convey to the reader: the CC is a nontrivial quantity in field theory *only* when there is a nontrivial metric. Let us first consider the classical field equations. The Bianchi identity fulfilled by the Einstein's tensor on the *l.h.s.* of (2), tells us that

$$\nabla^\mu \left(R^\nu_\mu - \frac{1}{2} R \, \delta^\nu_\mu \right) = 0 \,, \qquad \text{which implies} \qquad \nabla^\mu \left(8\pi G \, T^\nu_\mu + \Lambda \, \delta^\nu_\mu \right) = 0 \,. \tag{7}$$

Thus, if S_{HD} is negligible, G is constant and matter is covariantly conserved ($\nabla_\nu T^\nu_\mu = 0$), this requires Λ to be a constant (i.e. a general coordinate-invariant expression). Let us also notice that, if matter is non-covariantly conserved, equation (7) can still be satisfied through a special local conservation law that enforces transfer of energy between matter and the vacuum energy density (4). Indeed, using the FLRW metric, characterized by the scale factor $a = a(t)$, it is easy to see that the following relation holds [15]:

$$\frac{d}{dt} \left[G(\rho_\Lambda + \rho) \right] + 3 \, G \, H \, (\rho + p) = 0 \,, \tag{8}$$

where $H = \dot{a}/a$ is the expansion rate or Hubble function. At this point, there are still some non-trivial cosmological scenarios worth noticing. For example, for constant G the equation above yields $\dot{\rho}_\Lambda + \dot{\rho} + 3H(\rho + p) = 0$, which implies matter non-conservation, whereas if G is variable and matter is covariantly conserved, i.e. $\dot{\rho} + 3H(\rho + p) = 0$, then $(\rho + \rho_\Lambda)\dot{G} + G\dot{\rho}_\Lambda = 0$. It should be noticed that, in all these cases, we have the possibility of a time-evolving cosmological term, $\dot{\rho}_\Lambda \neq 0$.

However, a practical issue is: do we have enough information to solve the cosmological equations in these variable Λ scenarios? In other words, can we solve for the fundamental set of variables $H(t)$, $\rho(t)$, $p(t)$, $G(t)$, $\rho_\Lambda(t)$? If we take any of these models and combine the corresponding covariant conservation laws with Friedmann's equation

$$H^2 = \frac{8\pi \, G}{3} \left(\rho + \rho_\Lambda \right) - \frac{k}{a^2} \,, \tag{9}$$

it turns out that we cannot solve the cosmological problem yet. For, apart from the matter equation of state relating p and ρ, i.e. $p = \omega_m \rho$, we still need an additional equation to solve for the fundamental set of variables. Usually, the additional equation is provided in the literature on a purely *ad hoc* basis, see e.g. [25] and the numerous references therein. In these kind of phenomenological frameworks, one usually starts with a time evolution equation of the form

$$\frac{d\rho_\Lambda}{dt} = F(H, \rho, \rho_\Lambda, ...)\,, \tag{10}$$

in which F is some prescribed functional of the cosmological parameters. A prototype approach of this kind was considered e.g. in [26][3].

However, it is possible to motivate the needed equation for the CC evolution in more fundamental terms, namely from the QFT point of view, and more specifically from the renormalization group (RG) method [8]-[16]. In this case, the aim is to provide a differential relation (renormalization group equation) of the form

$$\frac{d\rho_\Lambda}{d\ln\mu} = \beta_\Lambda(P, \mu)\,, \tag{11}$$

where β_Λ is a function of the parameters P of the general action (1) and μ is a dimensional scale. The appearance of this arbitrary mass scale is characteristic of the renormalization procedure in QFT owing to the breaking of scale invariance by quantum effects. The quantity ρ_Λ in (11) is a (μ-dependent) renormalized part of the complete QFT structure of the CC. Depending on the renormalization scheme, the scale μ can have a more or less transparent physical meaning, but the physics should be completely independent of it. Such (overall) μ-independence of the observable quantities is actually the main message of the RG; but, remarkably enough, this same message also tells us that the RG can help to uncover the quantum effects precisely by focusing on the various μ-dependent pieces satisfying renormalization group equations like (11). Clearly, equations (10) and (11) are of very different nature. We shall further dwell on the (less obvious) physical interpretation of (11) in subsequent sections.

Essential for the RG method in cosmology is to understand that, in order for the vacuum energy to acquire dynamical properties, we need an evolving external metric background. For instance, for a spatially flat FLRW Universe (i.e. $k = 0$ in equation (9)) in the matter dominated epoch, the curvature scalar is given by

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = -12\,H^2 - 6\,\dot{H} = -3\,\Lambda - 3\,H^2\,. \tag{12}$$

It is, thus, indispensable to have a dynamical background, i.e. $H = H(t)$, in order to have an evolving 4-dimensional curvature. This dynamics (e.g. the observed accelerated expansion) is basically determined (at low energy) by Einstein's equations (2), but in general we may have the concurrence of the remaining terms in the extended action (1). And, what is more, we also have the quantum contributions, which can be highly non-trivial and play a crucial role.

In the absence of a dynamical background, all quantum effects are static and the RG equation (11) does not furnish any non-trivial information, in the sense that the observed CC value remains exactly the same at all times. In such situation, there remains no further μ-dependence

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[3]For recent developments on general models with variable cosmological parameters, see [27]

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apart from the trivial one associated to the static contributions. Therefore, an equation like (11) is still mathematically possible (cf. section 4), but we should not be misled by its appearance since it does not reflect any dynamical effect: it only says that $\rho_\Lambda = \rho_\Lambda(\mu)$ is a static quantity that can be renormalized at any value of μ in some arbitrarily chosen renormalization scheme.

In the general case, the observed CC is obtained from several parts $\rho_\Lambda^{(i)}(\mu)$ of the renormalized EA, namely from the original vacuum parameter $\rho_\Lambda^{\rm vac}(\mu)$, the possible induced effects $\rho_\Lambda^{\rm ind}(\mu)$ from spontaneous symmetry breaking (cf. section 4) as well as the contribution from quantum corrections of different kinds, some of them merely static and some others associated to the expanding cosmological background. A specific class of these quantum effects is, ultimately, the potential source of the genuine running (i.e. of the scaling) of the observed CC with some physical quantity (cf. section 5). Once the various μ-dependent parts have been (presumably) been identified with the help of the RG method, the final value of the CC is generated from the sum of all of them:

$$\rho_\Lambda^{\rm ph} = \sum_i \rho_\Lambda^{(i)}(\mu)\,. \tag{13}$$

If this sum could be theoretically computed in QFT in curved apace-time, it would represent the predicted physical observable associated to the CC, and could be confronted with the experimentally determined value $\rho_\Lambda^{\rm exp} \sim 10^{-47}\,GeV^4$. The number (13) should be free from all kinds of ambiguities and, in particular, μ-independent. As for the study of the scaling properties of this observable, the main point is that not all of the CC parts of the EA contribute to the physical scaling with, say, a momentum or the value of an external field (e.g. a non-trivial metric background). Therefore, the first task to do is to try to elucidate these especially relevant parts with the help of the RG. Some clues are given in the next section.

2.3   What means RG running in Quantum Field Theory

The properties of the classical theory depend on the dynamical equations, which can be defined via the principle of minimal action. So, within the current paradigm, we can state that the definition of a classical theory starts from establishing its action.

If the asymptotic states can be defined, the quantum theory can be characterized by the S-matrix elements. This is the usual situation in Particle Physics, for example. However, in gravity, this is not generally possible and then the definition of the S-matrix is problematic. In this context, the most useful approach to QFT is accomplished through the notion of effective action (EA), which is a functional of the mean fields (see, e.g., [33, 34, 35]). For a QFT with a single scalar field ϕ and Lagrangian density $\mathcal{L}$, the EA is given by the functional Legendre transform

$$\Gamma[\phi_c] = W[J] - \int d^4x\,\sqrt{-g}\,J(x)\,\phi_c(x)\,, \tag{14}$$

where ϕ_c is the classical or mean field. Here W is a functional of the source (which in turn is a functional of the mean field), and is defined through

$$e^{iW[J]} = \int \mathcal{D}\phi\; e^{iS_{scal}[\phi;J]} \equiv \int \mathcal{D}\phi\, \exp\left\{i \int d^4x\sqrt{-g}\; (\mathcal{L} + J\phi)\right\}\,, \tag{15}$$

where $S_{scal}[\phi;J]$ is the classical action of the scalar field in the presence of the source J. The mean field and the source are Legendre conjugate variables: $\phi_c = \delta W[J]/\delta J$. Moreover, taking

functional differentiations of the EA with respect to ϕ_c, one arrives at the one-particle irreducible Green's functions

$$\Gamma^{(n)}(x_1, ..., x_n) = \frac{\delta^n \Gamma[\phi_c]}{\delta\phi_c(x_1)...\delta\phi_c(x_1)}, \tag{16}$$

and, finally, to the amplitudes through the reduction formulae [34]. At the same time, the EA can be seen as a generalization of the classical action for the quantum domain. It is important to remember that, in contrast to the classical action, the EA always has certain ambiguities, which eventually disappear in the amplitudes. These ambiguities come from the choice of the parametrization (or gauge fixing, as a particular case) of the quantum fields and from the subtraction point associated to the choice of the renormalization scheme.

The derivation of the EA and working out its ambiguities can be regarded as the main target of QFT. In the perturbative approach, one arrives at the corrections ("quantum effects") to the classical theory in the form of a power series in the couplings. In practice, except for some special theories, one can not complete the task formulated above. The point is that quantum corrections are, typically, nonlocal expressions which are, at the same time, non-polynomial in the mean fields. There is, however, a special sector of the EA where the calculations can be accomplished. In renormalizable theories, the divergencies of the EA are given by local expressions which have the same functional dependence on the mean field as the terms of the classical action have on the classical fields. These divergencies are eliminated by adding the local counterterms. After that, one introduces the renormalized classical action, in which all fields and parameters are functions of an arbitrary renormalization scale μ_R (or subtraction point). Let us clarify that this is a general definition that applies to any renormalization scheme, so μ_R here is not meant necessarily to be the MS mass unit (see below). In more physical renormalization schemes (for instance, in momentum subtraction schemes), μ_R can be a momentum.

The next crucial step is to perform the fundamental identification between the bare and renormalized theories. To this end, one defines the renormalized EA to be equal to the bare EA:

$$\Gamma[g_{\alpha\beta}, \Phi(\mu_R), P(\mu_R), \mu_R] = \Gamma_0[g_{\alpha\beta}, \Phi_0, P_0], \tag{17}$$

where Φ_0 and P_0 stand for the full sets of bare fields and bare parameters, and $\Phi(\mu_R)$ and $P(\mu_R)$ are the renormalized ones. As usual, we have the renormalization transformations between bare and renormalized quantities, $\Phi_0 = Z_\Phi^{1/2}\Phi(\mu_R)$ and $P_0 = P(\mu_R) + \delta P(\mu_R)$, where the counterterms are to be fixed by some set of renormalization conditions (the MS ones being the simplest from the mathematical point of view). The *identity* (17) implies that the renormalized action satisfies the renormalization group equation

$$\mu_R \frac{d}{d\mu_R} \Gamma[g_{\alpha\beta}, \Phi(\mu_R), P(\mu_R), \mu_R] = 0. \tag{18}$$

Although this is a fundamental result, there is no need to prove it, as some authors curiously attempted to do in a recent paper [19], since it is true by definition! The identification (17) and corresponding RG equation (18), holds for both flat and curved space time, and for all kinds of renormalizable theories. Writing out the total derivative with the help of the chain rule one may cast (18) in the usual form of a partial differential equation (PDE) satisfied by the EA [4],

$$\left\{ \mu_R \frac{\partial}{\partial\mu_R} + \beta_P \frac{\partial}{\partial P} + \gamma_\Phi \int d^4x \sqrt{-g} \, \Phi \frac{\delta}{\delta\Phi} \right\} \Gamma[g_{\alpha\beta}, P(\mu_R), \Phi(\mu_R), \mu_R] = 0, \tag{19}$$

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[4]Here we disregard the classical dimension of the parameters.

where an implicit sum over parameters and fields is understood on the *l.h.s.* of (19). In this expression, we have the β_P-functions for all the renormalized parameters of the theory, and also the γ_Φ-functions (anomalous dimensions) for all the matter (mean) fields Φ (with wave function renormalization constants $Z_\Phi^{1/2}$). They are defined in the usual way:

$$\beta_P = \mu_R \frac{\partial P}{\partial \mu_R}\,, \qquad \gamma_\Phi = \mu_R \frac{\partial \ln Z_\Phi^{1/2}}{\partial \mu_R}\,. \tag{20}$$

The characteristics of the PDE (19) are well-known to be the running charges $P(\tau)$ for each renormalized parameter, with $\tau = \ln(\mu_R'/\mu_R)$.

Let us now concentrate for a while on the MS-scheme based renormalization group with dimensional regularization, as this is the framework which is more often used in practice, but at the same time the one whose physical interpretation is less obvious. Here the renormalization scale is the floating 't Hooft mass unit μ. For instance, the dimensionally regularized vacuum part of the action (1) in flat space-time boils down to just the bare CC term,

$$S_\Lambda = -\int d^n x\, \rho_\Lambda^0 = -\mu^{n-4} \int d^n x\, (\rho_\Lambda^{\mathrm{vac}} + \delta\rho_\Lambda^{\mathrm{vac}})\,, \tag{21}$$

where we have decomposed ρ_Λ^0 (the bare vacuum density in n-dimensions) into the renormalized one $\rho_\Lambda^{\mathrm{vac}} = \rho_\Lambda^{\mathrm{vac}}(\mu)$ plus a counterterm. The MS counterterm $\delta\rho_\Lambda^{\mathrm{vac}}$ will be fixed in section 4. At the moment, this simple example is only to illustrate that the artificial mass unit μ is introduced to make the regularized action dimensionless in such a way that the renormalized energy density $\rho_\Lambda^{\mathrm{vac}}(\mu)$ can stay 4-dimensional [5]. In general, the dimensional regularization scale μ allows that all parameters keep their original dimensions independent of the regularor n.

Similarly, we proceed to make the fundamental identification of the general form (17), but in this case for the particular MS scheme in dimensional regularization:

$$\Gamma[g_{\alpha\beta}, \Phi(\mu), P(\mu), n, \mu] = \Gamma_0[g_{\alpha\beta}, \Phi_0, P_0]\,. \tag{22}$$

Again, the overall MS-renormalized effective action Γ on the *l.h.s* of this equation does not depend on the floating mass scale μ. In other words, despite the non-trivial functional μ-dependence that the various terms of the EA may exhibit (see, for instance, the QED one-loop example (24) below), the full renormalized effective action has a value which does not depend on the particular numerical choice that we make for μ. It means, of course, that it satisfies a RG equation similar to (18), where in this case μ_R in that equation is replaced by the 't Hooft mass unit μ.

Let us repeat that the μ-dependence, even though artificial, manifests itself in the following two important ways:

• *Type-1* μ-dependence: All renormalized fields and parameters depend on μ, i.e. $\Phi = \Phi(\mu)$ and $P = P(\mu)$. This leads to an implicit μ-dependence of the EA;

• *Type-2* μ-dependence: The functional form of the EA depends explicitly on μ, as also indicated on the *l.h.s.* of (22).

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[5]The scale μ is the only arbitrary mass unit present in the MS scheme with dimensional regularization. However, the latter regularization can also be used in more physical schemes, like in the momentum subtraction scheme, and then both units μ and μ_R are present, where $|q| = \mu_R$ is the arbitrary momentum subtraction point.

The combination of these two types of μ-dependencies always cancels perfectly in the EA, i.e. we have the symbolic annihilation law[6]

$$\text{`` Type-1 + Type-2 = 0''} \quad \Longleftrightarrow \quad (\text{``}\mu\text{-annihilation''}) \quad \Longleftrightarrow \quad (\text{RG-invariance}). \quad (23)$$

This statement is true to any loop order, for otherwise the renormalized EA would be μ-dependent, violating its own definition (22). So, if we take together the two types of μ-dependencies, we just eliminate entirely the possibility to use it as a tool to explore the structure of the quantum corrections – see the QED case (24) studied below. The use of the RG, therefore, *requires* that the μ-dependencies of *Type-1* and *Type-2* are used separately. All these considerations are of course very well-known by the experts in the field; we are elaborating on them here because they are at the very root of some recent misuses of the RG in the literature, particularly in the cosmological context [19] (see section 4).

The property (23) is of course valid in any RG framework. However, in the MS-based scheme, where the renormalized quantities are simply defined by removing the poles (sometimes along with some finite terms) of the dimensionally regularized ones, it can be more difficult to interpret the physical sense of the formers. Moreover, being the 't Hooft mass unit μ such an artificial parameter, its physical interpretation always requires an additional effort. In QFT in flat space-time, we meet two different standard interpretations and each of them implies some relation between the MS scheme and another, more physical, renormalization scheme in the corresponding limit. Namely, for the μ-dependence of the *Type-1*, we have to seek correspondence with the momentum subtraction scheme (in particular, the on-shell scheme whenever possible), which is most useful one for the analysis of the scattering amplitudes at high energies [34]. In contrast, the μ-dependence of *Type-2* is mainly used for the analysis of phase transitions. Here we associate μ with the mean value of the almost static (usually scalar) field and arrive at the interpretation of the μ-dependent effective potential and its applications [37].

Once the divergencies are removed by the renormalization transformation of fields and parameters, as previously indicated, the MS-renormalized EA becomes finite. However, the different parts of the EA became dependent on the arbitrary renormalization parameter μ. This dependency shows up only in those parts of the EA which are related to divergences. For example, in massless QED (which, in practice, means in the limit $|q^2| \gg m^2$), the electromagnetic part of the one-loop renormalized effective action has the form

$$\Gamma_{em}^{(1)} = -\frac{1}{4e^2(\mu)} \int d^4 x \, F_{\mu\nu} \left[1 - \frac{e^2(\mu)}{12\pi} \ln\left(-\frac{\Box}{\mu^2}\right) \right] F^{\mu\nu}, \quad (24)$$

where $e(\mu)$ is the renormalized QED charge in the MS scheme. It is apparent from (24) that the effective (or "running") QED charge in momentum space satisfies

$$\frac{1}{e^2(q^2)} = \frac{1}{e^2(\mu^2)} - \frac{1}{12\pi^2} \ln\left(\frac{|q^2|}{\mu^2}\right), \qquad (|q^2| \gg m_e^2). \quad (25)$$

This relation also follows upon integrating the differential equation

$$\frac{d\,e(\mu)}{d\ln\mu} = \frac{e^3}{12\,\pi^2} \equiv \beta_e^{(1)} \quad (26)$$

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[6]Notice that, in the case of the S-matrix, the situation is similar, except that there are no μ-dependencies from the fields $\Phi(\mu)$ because the S-matrix does not depend at all on them.

from μ to μ' and then replacing $\mu'^2 \rightarrow |q^2|$. Here $\beta_e^{(1)}$ is the β-function of QED at one-loop in the MS. Equation (26) is correspondingly called the RG equation of the renormalized charge in the MS scheme. It resembles equation (11) for the Λ term (on which we will return later). Notice from (25) that $e(q^2)$ increases with $|q^2|$, as expected from the non-asymptotically free character of QED – a well tested feature of this theory.

The remarkable property of tracking explicitly the μ-dependence of the various parts of the EA is that the high-energy limit of the theory can be correctly reproduced, in the leading approximation, by taking the limit of large $\mu \gg m$. It is understood, when performing this limit, that all of the degrees of freedom, say N_f, having e.m. charge e and whose masses satisfy $m_f < \mu$, are included in the calculation of the β_e-function. In practice, this means that we can set the correspondence $\mu \rightarrow |q'|$, and in this way the QED running charge at the energy scale $|q|$ is related to the corresponding value at the other high energy scale $|q'|$ through

$$e^2(q'^2) = \frac{e^2(q^2)}{1 - N_f \dfrac{e^2(q^2)}{12\,\pi^2} \ln \dfrac{|q'^2|}{|q^2|}} \,, \tag{27}$$

a relation which, by the way, can be tested experimentally since it depends on physical quantities only [36].

Due to the simplicity of the form factor for the massless case (the above expressions are a nice illustration) one can always restore such form factor using the μ-dependence, i.e. the solution of the RG equation (26) in the MS scheme and the correspondence of μ with a high energy momentum. Let us clarify, however, that in QED there is no need to use the MS, if one does not wish to, because we may naturally proceed in the on-shell scheme, where the renormalization can be performed by subtracting the Green's functions on the mass-shells of the physical particles. Nevertheless, the lessons that we can learn from this simple QED example in the MS scheme are very important and should be stressed, to wit:

Lesson **i)**. The presence of the dimensional parameter μ is the characteristic sign of the dynamical breaking of scale invariance in QFT. As a result, the μ-dependence of the EA (or of the scattering matrix, if it exists) can be used in practice as a parametrization of the different quantum effects, despite that the EA and the scattering matrix are, overall, μ-independent. Thanks to this μ-parametrization provided by the MS-scheme based RG, we can tag the different types of quantum effects that the EA contains and this provides a clue for their identification, classification and evaluation.

Lesson **ii)**. If the scattering matrix can be defined, then at high energy the μ-dependence of the MS-renormalized amplitudes (S-matrix elements) can be translated into momentum dependence, and this leads to the notion of running charges with momentum, e.g. $e = e(q^2)$ in QED.

Keeping in mind these lessons is important, because it is often not possible to use a physical renormalization scheme. In Particle Physics, we happen to meet the two situations, namely we have QED and the electroweak standard model ($SU(2)_L \times U_Y(1)$) as excellent prototypes of theories where the on-shell scheme is ideally applicable, but we also have the archetypal example of quantum field theory where an off-shell renormalization scheme is mandatory: QCD, the $SU(3)_c$ gauge theory of strong interactions. In practice, this means that in QCD we must use some off-shell momentum subtraction scheme, or most often, simply the MS scheme [34, 35]. It

turns out that, in the gravity framework, we have a kind of similar (actually more complicated) situation when dealing with the CC parameter, in the sense that it is technically impossible at the moment to examine the renormalization of Λ in a physical subtraction scheme, even though it would be most desirable. As a result, we are temporally forced to use the MS scheme and examine the μ-dependence of the various terms that contribute to the vacuum energy density, as a modest tool to dig out the possible quantum effects of the theory, in particular the scaling properties of the CC (see sections 4 and 5 for more details).

In the massive case, the situation is more complicated. In the high energy limit there is an effective coincidence with the MS-based result, so the μ-dependence provides reliable information. At the same time, the low-energy limit has, typically, no direct relation to the μ-dependence. This is well-known in QED, for example, where the low energy limit of the vacuum polarization tensor is characterized by a behavior of the form $\Pi(q^2) \sim q^2/m^2$ [34]. Indeed, in this regime, the masses of the quantum fields dominate in the internal lines of the loops and in many cases we observe the quadratic decoupling, according to the standard Appelquist and Carazzone theorem [38]. In such situation, one has to be very careful in applying the MS-based results. In particular, the μ-dependence is no longer a good parametrization of these effects. The simplest option is to assume a "sharp cut-off", that means to disregard completely the contributions of massive fields at the energy scale below their mass ($|q^2| < m^2$) and, at the same time, treat their contributions above the proper mass scale ($|q^2| > m^2$) as high-energy ones, without taking decoupling effects into account. This procedure has been reflected in our QED example above. Let us notice that this was also the precise recipe adopted in our first articles on the CC running [8, 9], while in subsequent papers on the subject [10, 12, 15] the starting hypothesis was a quadratic decoupling. We shall discuss both options in more detail in the next sections.

At this point, it is appropriate to remark that *Lesson* **ii)** above, although it is of invaluable help in Particle Physics, it has a limited scope for the gravitational applications. In practice, due to the simplicity of the MS scheme, one often tries to use it as a heuristic approach to extract information on the radiative corrections. However, the basic problem that we have here is that, in the presence of a non-vanishing Λ, the space-time can not be flat – as it is transparent from Einstein's equations (2). Notice that even if the ρ_Λ^0 term in (1) would be set to zero, it is impossible to have a QFT in strict flat space-time since Einstein's equations could not be satisfied. In fact, take the simplest case of a free scalar field. The closed one-loop diagrams would generate the following (bare) vacuum energy density contribution in dimensional regularization:

$$\bar{V}_{vac}^{(1)} = \sum_k \frac{1}{2}\hbar\,\omega_k \equiv \frac{1}{2}\,\mu^{4-n} \int \frac{d^{n-1}k}{(2\pi)^{n-1}}\,\hbar\,\sqrt{\mathbf{k}^2 + m^2}\,. \tag{28}$$

Here we have included $\hbar$ for a while, just to emphasize that (28) acts as an effective CC induced by the quantum theory. In order to satisfy Einstein's equations, the space-time geometry on the *l.h.s.* of Eq. (2) is bound to become curved appropriately as a backreaction to the energy input on its *r.h.s.*. Furthermore, the subsequent renormalization of this QFT contribution requires to have the bare term ρ_Λ^0 back in the original action (see section 4). So, within the context of QFT, the geometry of the space-time can never be flat, strictly speaking.

The previous remark is important, for if one uses the MS scheme to renormalize the action (1) the resulting μ-dependence of the various terms can be hardly associated to any q-dependence

of a scattering amplitude (in contrast to QED or QCD). The reason is that the S-matrix cannot be properly defined in a curved background. Furthermore, except for the simplest static spaces, the definition of the effective potential problem is not easy in curved spaces [39]. And, finally, in curved space the existence of an almost static scalar field does not imply, in general, a static metric or static curvature. This is typical of a theory where two fields are present, in which the heavy one can be a scalar and the massless one represents the metric excitations.

Notwithstanding the limited applicability of *Lesson* **ii)** in the gravitational domain, let us emphasize that *Lesson* **i)** is, in contrast, still perfectly valid. As a matter of fact, it will be our main guiding paradigm in our search for the ultimate physical sources of the CC running within QFT in curved space-time. It actually gives the clue for the interpretation of the RG equation (11) for Λ. Namely, by integrating this equation with respect to μ, one obtains the functional μ-dependence $\rho_\Lambda = \rho_\Lambda(\mu)$ associated to that particular part of the EA contributing to the observed CC, and by an appropriate choice of μ in the cosmological context one may estimate the numerical contribution from this part.

Let us insist that there is no such thing as an RG equation (11) for the full physical CC, since the latter – being an observable quantity – does not depend on μ, cf. Eq. (13)! This trivial, but relevant, observation shows the radical distinction between the kind of equations (10) and (11) introduced in section 2. In short, the renormalization group Eq. (11) can only be applied to some CC parts of the EA which are parameterized by μ. The real problem here, of course, is that we don't know all the parts of the EA that feed the entire physical CC (most conspicuously the dynamical ones associated to the expanding background). Therefore, the particular choice of μ (for example, $\mu = H$, $1/a$, etc., see the next sections) can be relevant for the numerical estimate of the contribution from a concrete μ-dependent part that is known.

3 Renormalization group for Λ: brief history

Let us present a brief - most likely incomplete - list of the publications devoted to the renormalization group for the CC. A similar classification has been considered in a recent general review on the CC problem [40], but our comments may be somewhat different from the ones given in that review and are concentrated almost exclusively on the RG approach.

We start our list from the papers that treated the renormalization group for the CC in a mathematically consistent way, but did not consider cosmological applications. For example, in the pioneering paper by Nelson and Panangaden [41], the renormalization group for the CC and other parameters of the vacuum action was first linked to the global scaling of the metric. The same ideas were developed in more formal works by Buchbinder [42] and Toms and Parker [43]. It is also worth mentioning the first practical calculations [44]. A consistent pedagogical account of the renormalization group in curved space, together with much more references on the subject, can be found in the book [23] – the review of some recent achievements has been presented in [24]. In the above mentioned papers, gravity was treated as a classical background and only matter fields were quantized.

Other important works where the renormalization group for the CC has been considered are the papers by Salam and Strathdee [45] and by Fradkin and Tseytlin [46] (see also the subsequent calculations of [47, 48]). The object of quantization in these papers was higher derivative quantum gravity, and its renormalizability enables one to use the standard MS-scheme

based renormalization group for Λ, G and other parameters. Let us note, for the sake of completeness, that [46] includes also an alternative (on shell) approach to the renormalization group equation for Λ, which can be applied even to the non-renormalizable quantum General Relativity.

3.1 AI-type approaches

In the 1982 review paper [49], Polyakov made the intriguing observation that the IR effects of the unknown quantum gravity theory might lead to the effective screening of the CC at the cosmic scale. We can consider this remark as a starting point of the *AI* approaches (following the notation of section 1). In 1990 Taylor and Veneziano discussed the renormalization group as an instrument for solving the CC problem [50]. The main difficulty with these approaches is that no working model leading to the CC suppression in the infrared (IR) has been found.

The first such model traces back to 1992, which is when Antoniadis and Mottola introduced the idea that the "IR quantum gravity" could solve the CC problem through the mechanism of RG screening [51] (see also [52] for further developments on this proposal, and [53] for more recent discussions). This model of quantum gravity has been suggested earlier in [54], it implies that only the conformal factor of the metric should be quantized and that the starting action must be the one induced by the conformal anomaly [55]. The effective IR screening really takes place in this model, so in this sense it is very successful. However, it is by far not complete, because there are three unclear aspects, namely: *(i)* It is implicitly assumed that the IR decoupling of the massive mode of the conformal factor does not occur at low energies; *(ii)* Why we should quantize only the conformal factor of the metric? In particular, the problem of higher derivative ghosts is present in the "IR quantum gravity", exactly as in the usual higher derivative quantum gravity [56]; *(iii)* The ambiguities which are typical for the RG for CC in the higher derivative quantum gravity [46], should also be present in the theory of [51]. All three points are difficult to address and this has been never done in a comprehensive way, despite that an attempt to deal with the item *(i)* has been done in [57].

Another two realizations of the "IR quantum gravity" and the screening of CC can be attributed to the works [58] and [59]. In the first case, the central idea is to assume the existence of the non-Gaussian fixed point in quantum gravity. After that, the existing ambiguity in the non-perturbative RG equations for the CC and the Newton constant G is fixed by requiring that G tends to the constant value in the IR limit. As a result, we observe a screening of the CC at low energies. Let us remark that exactly the same idea has been published earlier in the paper [61], which belongs to the *AII* class of approaches and will be discussed in the next subsection. In an alternative approach which was used in [59], the IR screening of the CC is derived without the explicit use of RG. The calculations are based on the original coordinate-representation Feynman diagrams in the deSitter space (dS). Unfortunately the calculations on dS do not distinguish the CC, Einstein-Hilbert term and the higher derivative terms in the vacuum action, while in reality all of them have their own RG [24] and hence their own leading-log corrections. Therefore it is unclear what is actually screened in this model. This shortcoming is indeed still present in the last version on the model [60], despite it is stated that this approach is more correct than the RG one.

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3.2 AII and BII-type approaches

The first model with the IR screening for the CC due to the quantum effects of matter fields was constructed by one of the present authors in [61] (see also further discussion in [5] and similar model suggested by Jackiw et al [62]). Let us remind that the observable CC consists of the vacuum and induced parts. The vacuum CC is necessary in the quantum field theory, since it provides renormalizability in the vacuum sector (see [9] for a detailed discussion). However, if the theory is completely massless, one can avoid introducing the vacuum CC and then one needs to have the screening for the induced component only. The main assumption in [61] was that the origin of all masses is the Coleman-Weinberg mechanism (dimensional transmutation) within some GUT model, where the scalar field ϕ is non-minimally coupled to curvature (i.e. $\sim \xi \phi^2 R$) and has self-interaction ($\sim f \phi^4$). The induced quantities are, in this case,

$$(16\pi G)^{-1} \ \sim \ <\xi\phi^2> \qquad \text{and} \qquad \rho_\Lambda \ \sim \ <f\phi^4>, \tag{29}$$

The RG behavior of these induced quantities depend on the RG equations for ϕ, f, ξ. Fixing the usual ambiguity in the γ-function for ϕ such that $G \to const$ in the IR, for a wide class of GUT models we arrive at the effective screening of the CC,

$$\rho_\Lambda \ \sim \ (\mu/\mu_0)^{2|A|g^2} \longrightarrow 0\,, \tag{30}$$

where μ and μ_0 are the current cosmic energy scale and initial high energy scale, respectively.

Despite this model does not look phenomenologically realistic, it shows that the quantum effects may be relevant for solving the CC problem, so that it is worthwhile to work further in this direction.

The first *BII*-type model for the study of the CC running has been suggested by the present authors in [8, 9], where we used the standard MS-based RG for the CC [41, 42, 43][7]. The scale parameter μ has been associated to the energy scale defined by the critical density, i.e. $\mu \sim \rho_c^{1/4}$. During the hot stages of the evolution, say at temperature T, such scale obviously behaves as $\mu \sim T \sim a^{-1}$, where $a = a(t)$ is the scale factor, whereas in the current Universe it behaves as $\mu \sim \sqrt{H_0 M_P}$, i.e. as the geometric mean of the two extreme physical energy scales in the IR and the UV in our Universe. The use of the MS-scheme based RG was implemented through the "sharp cut-off" approximation (see section 2.3). In this case it means that, in the current Universe, we assumed that there is a sharp decoupling of all particles having masses m above the $\mu \sim \rho_c^{1/4} \sim \sqrt{H_0 M_P}$ energy scale. At the same time, following the MS-based RG prescription, we took into account the contribution from all the light degrees of freedom satisfying $m < \mu$ and considered their effect for the running of all the parameters, without bothering about their mass effects (as this is part of the MS prescription). Since the lightest neutrinos could have masses below the current value of the cosmic scale in this framework, i.e. $m_\nu \lesssim\sim \sqrt{H_0 M_P} \sim 10^{-3}\, eV$, our procedure took into account the quantum effects on the CC running from the lightest neutrinos, which are the natural candidates here from the SM, including some hypothetical light scalar fields which may emerge in the extensions of the SM (like axions and cosmons [63]). As a result, the naturally predicted value of the CC evolution in this range is $\Delta\rho_\Lambda \sim \mu^4 \sim H_0^2 M_P^2$, which is of the order of the CC value itself at the present

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[7]See e.g. [35] for a pedagogical introduction to the RG for the CC in the MS scheme.

time. The cosmological implications based on this RG scenario have been developed in [11, 64], and also recently in an analysis of the cosmological neutrino mass bounds in the presence of a running CC, see [65] [8].

The subsequent works on the *BII* approach have explored the effects of a smooth Appelquist and Carazzone - like decoupling and greatly benefited from the contributions of Babic, Guberina, Horvat and Štefančić [10, 14]. An interesting proposal was the possibility to use the Hubble scale "$\mu = H$" (cf. [9]) along the idea of "soft decoupling" of the GUT fields. This scenario provided a plausible phenomenological model for the physical running of the CC in a mass-dependent framework, and was amply explored in subsequent publications [12, 13, 15, 16, 17] [9]. It is based on the IR ansatz:

$$\rho_\Lambda(H) = \rho_\Lambda^0 + \frac{3\nu}{8\pi} \left(H^2 - H_0^2\right) M_P^2 + \mathcal{O}\left(H^4\right) + \mathcal{O}\left(\frac{H^6}{m^2}\right) + ...,\tag{31}$$

in which the leading term (the so-called "soft-decoupling" term) is quadratic in the expansion rate H, and the remaining powers of H are the ordinary decoupling contributions of the Appelquist-Carazzone type [38]. Here ν is a small parameter, essentially given by the ratio squared of the average mass M of the GUT particles to the Planck mass, specifically $\nu = (1/12\pi)M^2/M_P^2$. In Ref. [17], this parameter was stringently bounded ($\nu < 10^{-4}$) from the analysis of cosmological perturbations. Notice that, in this model, one also obtains that the typical amount of running is $\Delta\rho_\Lambda \sim H^2 M_P^2$, as in the previous one. However, in the present case, the concept is different, namely, the evolution of the CC is not $\sim m^4$ (with $m \lesssim \mu \sim \sqrt{H M_P}$), but of the soft decoupling type $q^2 M^2$, where $|q|^2 \sim |R| \sim H^2$ is the typical momentum squared of the cosmological gravitons – see (12) – and involves, in contrast, the heaviest degrees of freedom. The explicit appearance of masses and momenta makes this formulation more physical and the association "$\mu = H$" can be interpreted as a momentum subtraction scale, i.e. $\mu_R = |q| \simeq H$ (quite different from the MS case) along the lines of (17).

Let us clarify that the ansatz (31) refers to the effective IR behavior, assumed in these references, for the vacuum part of the CC only. In order to really build up constructively this expression using the RG method, we should go through the computation of the various quantum effects. Let us notice that this approach could emerge as a most natural description of the vacuum effective action at low energies [9, 18, 24], as we shall also discuss in section 5.

Finally, we have to mention the explicit derivation of the Appelquist and Carazzone theorem for gravity [67]. The net result of these calculations is that we can establish and prove this theorem for the higher derivative sector of the vacuum action, but not for the CC and Einstein-Hilbert terms. It is important to emphasize that this does not mean that there is no running in these sectors. The output of the calculations of [67] only shows that the currently available methods are not appropriate for the mentioned purpose. These methods are based on the flat space expansion and can not lead to the desired running of ρ_Λ and G independently of whether such running takes place or not.

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[8]For other interesting implications of the neutrino physics on the running of the CC, see [66].
[9]Let us note that the two scale associations $\mu \sim 1/a$ and $\mu \sim H$ could be related [18].

4 Effective potential and cosmological constant

The idea that the cosmological term could be adjusted dynamically first appeared, historically, within the context of the effective potential models of scalar field theories modeling the vacuum state [68, 63, 69]. More recently, the scalar field models have overflowed the "cosmological marked" in the form of quintessence, k-essence and the like [2] and have strengthened the perspective for a possible dynamical nature of the dark energy. Unfortunately, at the moment, all the proposed scalar fields that hypothetically could solve the CC problem are completely *ad hoc* and, in addition, none of them has any obvious relationship with the Particle Physics world, e.g. with the SM of the strong and electroweak interactions or with some promising extension of it (like the MSSM, or some favorite supersymmetric GUT [7]).

On the other hand, in the renormalization group context, a first useful proposal to describe the dynamical dark energy as a running CC was put forward in [8, 9] within the context of the SM of Particle Physics, and it made use of the properties of the effective potential of the SM (i.e. the Higgs potential). Later on, in [12, 13, 15, 16] various cosmological models have been developed on the basis of the possible running of the CC along the lines sketched in section 2.2 and, in particular, using the phenomenological ansatz (31).

Instead of thinking about the cosmological models with running Λ one may adopt, instead, an opposite attitude and try to prove that such running is mathematically impossible. An attempt in this direction has been undertaken in a recent article [19]. Unfortunately this short, simple and apparently clear work is plagued by a number of conceptual mistakes. After this preprint was submitted to the arXiv, several colleagues asked us to explain the situation [10]. So, correcting the misconceptions of [19] seems a necessary step to restore the truth.

The consideration of [19] is based on the renormalization group equation for the effective potential of the real massive scalar field with the $\lambda \phi^4$ interaction. The classical part of the potential is

$$U(\phi) = \frac{m^2 \phi^2}{2} + \frac{\lambda \phi^4}{4!} \,. \tag{32}$$

The quantum corrections are then computed in the MS scheme with dimensional regularization. One obtains an expression of the form

$$V_{eff}(\phi) = U(\phi) + \hbar V^{(1)}(\phi) + \hbar^2 V^{(2)}(\phi) + ...,, \tag{33}$$

where the powers of $\hbar$ count the number of loops. Later on in this section we shall consider the specific case of the one-loop correction.

This situation applies also to the Higgs potential of the SM, which was addressed in the early papers [8, 9]. In these papers, we defined (in some equivalent notation) the expression

$$\rho_\Lambda(\mu) = \rho_\Lambda^{\text{vac}}(\mu) + \rho_\Lambda^{\text{ind}}(\mu) \,, \tag{34}$$

which is the sum of the vacuum and induced parts of the CC. Of course, the quantity (34) is not intended to be the entire observable CC, i.e. Eq. (13), because the above sum is μ-dependent

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[10]Let us remark, in passing, that the version of this paper which has been made public had no essential difference with the one which these authors have sent to us prior submission. We have explained them what is the correct point of view, but the only effect of these explanations on them was to thank us for the correspondence in the Acknowledgments of [19].

whereas the observable CC is not! The two terms on the r.h.s. of this expression constitute only the, so-called, *Type-1* contributions (see section 2.3) to the vacuum and induced parts of the CC. In other words, they can be regarded as two independent contributions to the overall sum on the *r.h.s.* of Eq. (13). As we remarked in section 2.3, the RG method is based on tagging and classifying the two types of contributions (*Type-1* and *Type-2*) and it is essential not to mix them, otherwise the μ-dependence disappears altogether, as indicated in (23); in this case, we would lose the chance to pinpoint where are the possible parts of the EA connected with the physical running. Below we explain the meaning of (34) in more detail and we identify explicitly the two terms $\rho_\Lambda^{\rm vac}(\mu)$ and $\rho_\Lambda^{\rm ind}(\mu)$.

The vacuum CC for the above model can be included into the scalar potential in the form hm^4, where m is the scalar mass and h is an independent dimensionless parameter, similar to the parameter f in (6). Let us recall that the effective potential for local field theories is related to the effective action in the limit of constant mean field ($\phi = const.$) through

$$\Gamma[\phi] = -\int d^4x \sqrt{-g}\, V_{eff}(\phi, m^2, \lambda, h, \mu) = -\Omega\, V_{eff}(\phi, m^2, \lambda, h, \mu)\,, \tag{35}$$

where Ω is the space-time volume. Therefore, from the fundamental RG equation (19) satisfied by the renormalized effective action in the MS scheme with dimensional regularization ($\mu_R \rightarrow \mu$), we immediately find the corresponding RG equation satisfied by the renormalized effective potential of such theories:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + \gamma_m m^2\frac{\partial}{\partial m^2} + \beta_h\frac{\partial}{\partial h} + \gamma_\phi\phi\frac{\partial}{\partial\phi}\right) V_{eff}(\phi, m^2, \lambda, h, \mu) = 0\,, \tag{36}$$

where we have specified the contributions from the parameters $P = \lambda, m, h$, with corresponding β_P-functions (20) (the one for the mass squared is usually relabeled as $\beta_{m^2} \equiv \gamma_m\, m^2$). As remarked in section 2.3, there is nothing to be proven here; equation (36) is true just by definition of effective action and effective potential. As a next step, the authors of [19] notice that, in the minimum of the potential, $\partial V/\partial\phi = 0$, and therefore the value of the potential in the minimum is μ-independent[11]. Their conclusion (although obvious *ab initio!*) is that the minimal value of the overall effective potential is μ-independent. The aforementioned authors interpret this result as a kind of "non-running theorem", i.e. as a formal demonstration that the observed value of the vacuum energy does not run, in contrast to the conclusion of our original papers [8, 9]. The effect of other fields (e.g. the SM constituents), or the presence of an external gravitational background, does not change – according to the authors of [19] – this conclusion at all.

In the light of the discussion presented in the previous sections, specifically in 2.2 and 2.3, it should be vastly apparent by now that the main theses of Ref. [19] are completely unjustified (in fact, entirely wrong) as they are based on a (severe) conceptual mistake about the significance of the RG as a tool to explore the quantum effects in QFT. Let us also note that while these authors cite only our papers on the CC running, their "criticisms" go simultaneously at the heart of the sizeable body of respectable works on the CC running in the literature, many of which we have cited in section 3. In fact, most of them are also based on the MS-scheme of

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[11]One can immediately notice that in the $\lambda\phi^4$-theory the use of the condition $\partial V/\partial\phi = 0$ is not very relevant, because $\gamma_\phi = 0$ at one-loop.

renormalization[12]. For better clarity, let us try to classify in some more detail the incorrect misconceptions contained in Ref. [19] (which they used against our original works [8, 9]). The main reason for doing this is to try to "undo" as much as possible the chain of possible confusions that this work might have spread over some readers. It will also serve as a basis for a pedagogical discussion of the subject.

• First. Physical observables (in particular the observed CC obtained from supernovae, CMB and LSS data [4]) are RG-invariant (i.e. μ-independent) quantities. Therefore, trying to prove that a physical observable is RG-invariant is a (useless) tautology.

• Second. The essential confusion of these authors is to incorrectly identify the physical running of the cosmological term with the μ-dependence. This is obviously wrong and is connected with the previous point. One thing is the mathematical "running" of some parameters of the EA with the floating mass scale μ (i.e. what we called Type-1 μ-dependence in section 2.3), together with the explicit μ-dependence of some parts of the EA (Type-2 μ-dependence), and another (quite different) thing is the measurable scaling of a physical observable (in this case the observed CC) with a physical quantity, say a momentum q or an external field (such as e.g. the metric and its derivatives in the presence of a non-trivial background). While the μ-dependence can be used in some cases to trace the q-dependence (or the external field dependence) of the quantum corrections to the S-matrix (or to the EA), the μ-dependence in itself has no intrinsic physical meaning, because μ cancels in the overall result. The physical running (if it is there at all) is not in μ but in q (or in the dynamical properties of the external background metric). For example, in the model of Eq. (31), the physical running lies in the H-dependence of the result. This is of course true irrespective of whether the floating mass scale μ of the various pieces is explicitly kept or not, even though we know it must cancel in the complete CC expression. Still, it is customary (by an abuse of language) to refer to this μ-dependence of the renormalized quantities as a kind of "running" with μ. We have no problem in keeping this usage of words, but the reader should be careful in distinguishing it from the physical running or scaling of the parameters with momenta or field strengths.

• Third. Equation (36) is, in fact, a sum of two independent equations. Indeed, one can split the overall effective potential as a sum of two pieces, the ϕ-independent (vacuum) term and the ϕ-dependent (scalar) term, as follows:

$$V_{eff}(\phi, m^2, \lambda, h, \mu) = V_{scal}(\phi, m^2, \lambda, \mu) + V_{vac}(m^2, \lambda, h, \mu). \tag{37}$$

In this way we can also split the RG equation (36) into two independent RG identities:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + \gamma_m m^2 \frac{\partial}{\partial m^2} + \gamma_\phi \phi \frac{\partial}{\partial \phi} \right) V_{scal}(\phi, m^2, \lambda, \mu) = 0, \tag{38}$$

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + \gamma_m m^2 \frac{\partial}{\partial m^2} + \beta_h \frac{\partial}{\partial h} \right) V_{vac}(m^2, \lambda, h, \mu) = 0. \tag{39}$$

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[12]For example, the RG approach of our first paper [8] was exactly a standard MS-based one as in the works [41, 42, 43]. The important new element was the physical interpretation of it within the Particle Physics phenomenological framework, especially within the SM of strong and electroweak interactions.

In order to understand the origin of this splitting, one has to introduce the functional called effective action of vacuum. This object is that part of the overall EA which remains nonzero when the mean scalar field is set to zero. The vacuum EA comes from the Legendre transform of the generating functional of the one-particle irreducible Green's functions without the source term for the scalar field, i.e. $J = 0$ in equations (14),(15):

$$e^{i\Gamma_{vac}} = e^{iW_{vac}} = \int \mathcal{D}\phi \, e^{iS_{scal}[\phi; J=0]} \,. \tag{40}$$

In flat space-time, the functionals Γ_{vac} and W_{vac} are equal numbers which do not depend on the field variables; they are the generators of the proper vacuum-to-vacuum diagrams. In flat space-time, these functionals are usually disregarded as irrelevant constants, and moreover their divergences can be eliminated by changing the operator ordering. At the functional level, this is equivalent to normalize the functional (40) to one (using the arbitrary normalization prefactor that carries with it).

However, in the presence of a gravitational background, the situation can be non-trivial, in contrast to the claims of [19]. Due to the presence of an external metric, it is more advisable to take Γ_{vac} to be a subject of the renormalization procedure and remove its divergence by renormalizing the parameter h (for the sake of convenience of the reader, we keep some of the notations of [19] in this section). From the RG-invariance of the renormalized EA – see Eq. (19) – it follows immediately the μ-independence of the renormalized functionals W_{vac} and Γ_{vac} and, therefore, we arrive at the second identity (39) for the vacuum part of the effective potential, while the first identity is the result of the subtraction of (39) from (36).

The net result of these considerations is that the vacuum and matter parts of the EA and, consequently, of the effective potential, are μ-independent separately and no cancelation between them can be expected.

● Fourth. The quantity (34), which we used in [8, 9], may now be unambiguously defined; namely, it is just the sum of the respective *Type-1* μ-dependent parts of the two RG-invariant pieces, V_{scal} and V_{vac}, of the effective potential. The *Type-2* dependences from these parts of the effective potential are *not* included, otherwise the μ-annihilation relation (23) would hold. As a result, $\rho_\Lambda(\mu)$ in (34) is a neatly μ-dependent expression; specifically, it consists of the parts that we would usually put in correspondence with the momentum subtraction scheme (or even with the on-shell scheme), if that would be possible. But in those cases where the correspondence with a physical renormalization scheme is especially difficult and unclear, one may try the sharp cut-off approximation in the MS scheme; and this is precisely what we did in [8, 9].

If the "proof" of the no-running CC proposed in [19] would make sense at all, one could immediately extend its scope with major implications. Thus, one could easily "demonstrate" such no-running feature in all of the theories where the overall μ-dependence is absent, some of them being quite conspicuous by the way, such as the NJL model, QED and QCD. It is especially easy to do for the NJL model, because the difference with the CC case can be disposed of by just changing the parametrization of the external metric, see equation (6) and the corresponding discussion. Of course, proceeding in this manner would be a rather unproductive, in fact absurd, way of using the MS-scheme based RG. Nevertheless, it seems that it is precisely what these authors have done in their completely unsuccessful analysis of the CC problem.

• For a correct analysis, the two RG equations (4) and (39) for the split effective potential must be considered separately, and then we have to extract the corresponding *Type-1* μ-dependent parts in each of these equation. The first one contains information about the running of the ϕ-dependent part of the effective potential and, therefore, about the running of the induced CC [8], whereas the second one includes the necessary information on the running of an independent parameter – the vacuum CC part:

$$\rho_\Lambda^{\text{vac}} = \frac{\Lambda}{8\pi G} = hm^4 \,. \tag{41}$$

Let us start the analysis with this part. The classical expressions are related by a very simple relation

$$U_{vac} = \rho_\Lambda^{\text{vac}} \,. \tag{42}$$

So how can it be that $V_{vac}(m^2, \lambda, h, \mu)$ is identically μ-independent while $\rho_\Lambda^{\text{vac}}$ does "run" with μ? However, it is exactly the situation which takes place in the MS-scheme of renormalization. The whole point is that the relation (42) gets modified at the quantum level. For example, at one-loop we have, instead,

$$V_{vac} = \rho_\Lambda^{\text{vac}}(\mu) + \frac{1}{2}\beta_\Lambda^{(1)} \ln\left(\frac{\mu_0^2}{\mu^2}\right), \tag{43}$$

where $\beta_\Lambda^{(1)}$ is the one-loop β-function for the CC term. In order to keep V_{vac} identically μ-independent, we expect that the renormalized parameter $\rho_\Lambda^{\text{vac}}(\mu)$ "runs" with μ according to

$$\rho_\Lambda^{\text{vac}}(\mu) = \rho_\Lambda^{\text{vac}}(\mu_0) - \frac{1}{2}\beta_\Lambda^{(1)} \ln\left(\frac{\mu_0^2}{\mu^2}\right). \tag{44}$$

One can easily check that these relations hold in particular cases. For example, let us take a scalar field with mass m. If we compute explicitly the corresponding vacuum-to-vacuum diagram at one-loop in dimensional regularization, i.e. Eq. (28), and add up the result to the action (21), we easily find

$$V_{vac} = \rho_\Lambda^{\text{vac}}(\mu) + \delta\rho_\Lambda^{\text{vac}} + \bar{V}_{vac}^{(1)} \,, \tag{45}$$

where the unrenormalized one-loop correction reads

$$\bar{V}_{vac}^{(1)} = \frac{m^4}{64\,\pi^2}\left(-\frac{2}{4-n} - \ln\frac{4\pi\mu^2}{m^2} + \gamma_E - \frac{3}{2}\right). \tag{46}$$

Let us adopt the $\overline{MS}$ subtraction scheme, characterized by the counterterm

$$\delta\rho_\Lambda^{\text{vac}} = \frac{m^4}{64\,\pi^2}\left(\frac{2}{4-n} + \ln 4\pi - \gamma_E\right). \tag{47}$$

It follows that the $\overline{MS}$-renormalized vacuum part of the potential is

$$V_{vac}(m, \mu) = \rho_\Lambda^{\text{vac}}(\mu) + V_{vac}^{(1)}(\mu), \tag{48}$$

with the finite one-loop piece

$$V_{vac}^{(1)}(\mu) = \frac{m^4}{64\,\pi^2}\left(\ln\frac{m^2}{\mu^2} - \frac{3}{2}\right). \tag{49}$$

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Notice that, at one-loop, it does not depend on the self-coupling λ. Clearly, (48) has the form (43) and we can identify

$$\beta_\Lambda^{(1)} = \frac{m^4}{32\,\pi^2}\,. \tag{50}$$

Plugging next this β-function on the *r.h.s* of Eq. (11), with the understanding that ρ_Λ there is the $\rho_\Lambda^{\rm vac}$ part under consideration, we may easily integrate the RG equation for the vacuum CC. Using the fact that at one-loop the mass m does not run with μ, we arrive at the result

$$\rho_\Lambda^{\rm vac}(\mu) = \rho_\Lambda^{\rm vac}(\mu_0) - \frac{m^4}{64\,\pi^2}\,\ln\frac{\mu_0^2}{\mu^2}\,, \tag{51}$$

which confirms the general expectation (44).

Furthermore, we can check that after replacing Eq. (51) into (48) we obtain a formally identical expression in which μ has been replaced by μ_0. This is the realization, in this particular example, of the RG-invariance of the vacuum effective potential, i.e. of Eq. (39). Thus, the μ-dependence displayed on the *l.h.s.* of Eq. (48) is only to indicate that the various parts on the *r.h.s* of that expression are non-trivially parameterized by μ, but the sum of these parts is ultimately independent of it. The procedure can be extended to any loop order, and then $V_{vac}(m,\lambda)$ is in general a function of both the mass m and the self-coupling λ.

● The situation for the induced vacuum energy density is similar. The form of the renormalized effective potential in the $\overline{MS}$ scheme for the scalar field, with a classical potential $U(\phi)$, is well known (see, e.g., [34]). At one-loop, Eq. (33) reads (we set $\hbar = 1$ again)

$$V_{eff}(\phi) = U(\phi) + V^{(1)}(\phi) = U(\phi) + \frac{1}{64\pi^2}\,U''^2\left(\ln\frac{U''^2}{\mu^2} - \frac{3}{2}\right). \tag{52}$$

In the case of (32), the one-loop correction yields

$$V^{(1)}(\phi) = \frac{1}{64\pi^2}\left(m^2 + \frac{\lambda\phi^2}{2}\right)^2\left[\ln\frac{(m^2 + \lambda\phi^2/2)}{\mu^2} - \frac{3}{2}\right]. \tag{53}$$

To obtain $V_{scal}(\phi, m^2, \lambda, \mu)$, we just subtract from (52) the vacuum part at one-loop, which is given by the second term on the *r.h.s.* of (48):

$$V_{vac}^{(1)} = \frac{m^4}{64\pi^2}\left[\ln\frac{m^2}{\mu^2} - \frac{3}{2}\right] = V_{eff}(\phi = 0, m^2, \lambda, \mu)\,, \tag{54}$$

where the second equality reflects that the first expression can also be obtained by setting $\phi = 0$ in the full one-loop effective potential (52). Therefore, the ϕ-dependent part of the potential finally reads

$$V_{scal}(\phi, m^2, \lambda, \mu) = V_{eff}(\phi, m^2, \lambda, \mu) - V_{eff}(\phi = 0, m^2, \lambda, \mu)\,. \tag{55}$$

This expression satisfies Eq. (4). To understand why, let us first remark the existence of an explicit (i.e. *Type-2*) μ-dependence in the loop correction term (53). For a massless case, one can restore the effective potential from the μ-dependence alone.

The fact that the effective potential part (55) depends explicitly on μ and, at the same time, satisfies (4) is because there is still the μ-dependence that is associated to the renormalization

of the parameters λ and m (i.e. the *Type-1* μ-dependence). Let us indeed recall that, once the two types of μ-dependences meet together, they annihilate each other – see Eq. (23) [13].

For the sake of simplicity, we can illustrate the fulfillment of the RG-invariance of V_{scal} in the case when the mass m is negligible. Then, it is enough to consider the renormalization of the λ coupling, whose one-loop RG equation in the MS scheme has the form

$$\mu \frac{d\lambda}{d\mu} = \beta_\lambda^{(1)} = \frac{3\lambda^2}{(4\pi)^2}. \tag{56}$$

Integrating this equation from μ to μ', we obtain:

$$\lambda(\mu') = \frac{\lambda(\mu)}{1 - \frac{3\,\lambda(\mu)}{32\,\pi^2} \ln \frac{\mu'^2}{\mu^2}} \simeq \lambda(\mu) + \beta_\lambda^{(1)}\,\tau\,, \tag{57}$$

where $\tau = (1/2)\ln \mu'^2/\mu^2$. Notice that the obtained result is similar to (27) for the QED case. This exemplifies, once more, that the μ-dependence of this kind in the MS scheme is of *Type-1* and, hence, it can be associated to the momentum subtraction scheme in the high energy limit.

If we replace (57) into (53) – in the limit of negligible m – and disregard the $\mathcal{O}(\lambda^3)$ terms that can be affected by higher order corrections, the expression (55) is transformed into another one which is formally identical, but in which μ is traded for μ'. In other words, this confirms that V_{scal} is RG-invariant to the order under consideration, as we expected. But, again, we are not interested in this complete RG-invariant expression for the study of the possible sources of running. We, instead, focus only on the implicit (*Type-I*) μ-dependence of V_{scal}, which is obtained by replacing the parameter m and λ by the RG-improved ones $m(\tau)$ and $\lambda(\tau)$ at the minimum of (32). The result is

$$\rho_\Lambda^{\mathrm{ind}}(\tau) = -\frac{3\,m^4(\tau)}{2\lambda(\tau)}. \tag{58}$$

Let us point out that, in the context of the SM, with all the fermions and bosons involved, the running of $\lambda = \lambda(\tau)$ is not just given by (56). The complete expressions for the RG equations of the parameters λ and m within the SM are given in Ref. [8, 9].

As explained previously, in the cosmological analysis of these references, the correspondence of the MS with the high energy limit just means to take into account the correct number of degrees o freedom involved in the computation of the β-functions (for both the vacuum and induced terms in (34)). Namely, they should only involve the lightest neutrinos and other light degrees of freedom potentially present in the current Universe. In no way presumes to consider that the present Universe is in a high energy state! Put another way, the use of the MS scheme in the "high energy" limit means, in the present context, to consider only the contribution from particles whose masses masses satisfy $q^2 > m^2$, for q of order of $\mu \sim \sqrt{H_0\,M_P} \sim 10^{-3}\,eV$. Using (50) and (58), this leads, finally, to the RG equation for (34) at the present cosmic scale:

$$(4\pi)^2\,\frac{d\rho_\Lambda}{d\tau} = (4\pi)^2\,\left(\frac{d\rho_\Lambda^{\mathrm{vac}}}{d\tau} + \frac{d\rho_\Lambda^{\mathrm{ind}}}{d\tau}\right) = \frac{1}{2}\sum_s m_s^4 - 4\sum_\nu m_\nu^4\,, \tag{59}$$

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[13]We point out that, at one-loop, there is no μ-dependence associated to the field ϕ itself, at least for a scalar theory based on a the classical potential (32). This is a reflect of the fact that $\gamma_\phi = 0$ at this order in such theory.

where the sums extend over light scalars (e.g. the cosmon [63]) as well as over light neutrinos, all of them satisfying the previous sharp cutoff condition. The neutrino term in (59) receives contributions from both the vacuum and induced parts [8].

An essential observation about the cutoff $\mu \sim \sqrt{H_0 M_P}$ is that it is determined by the value of the expansion rate at the present time, $H_0 \sim 10^{-33} \, eV$, but of course the method can be iterated to values of H at any time in the history of the Universe, see the analysis of [9]. This fact clearly shows that, in the absence of an expanding background, the whole procedure ceases to make sense. This crucial aspect went also completely unnoticed by the authors of Ref. [19]. Despite its limitations, the H-dependence of the cutoff is an effective tool that the MS-scheme based RG possesses to explore the possible dependence of the EA on the external field. In more formal terms: the presence of the external metric is the actual source for the possible running. The relevant Feynman diagrams include matter fields loops with a number of external tails of the metric (see, e.g., [24]). In this situation, even if we can only rely upon the MS-renormalization scheme, there are many choices for a physically reasonable identification of μ. We used some possibilities in [8, 9] – one of them has just been mentioned above. However, at variance with the flat space-time case, there is not a single obvious choice comparable to the S-matrix problem in high energy physics, where μ can be taken of order of the typical energy of the scattered particle.

● The observed CC should emerge from the complete (and RG-invariant) theoretical expression (13), provided it would be really feasible to explicitly account for it (i.e. if it would be possible to solve the "old CC problem" in some concrete TOE framework! – see section 1). However, accomplishing this final aim, is well out of the scope of the RG analysis, just because of the RG-invariance of the CC as a physical observable! Let us recall the reader, once more, that the RG considerations are framed in the context of the *BII* kind of approach to the CC problem. Therefore, the partial sum (34) should suffice to fulfil the (much more modest) RG aim since it already carries the real "signature" of the possible running, namely it includes the *Type-I* μ-dependence. Why only this μ-dependence? Because *the difference with the complete expression does not matter for the running!* The dependence of the EA on the metric and its derivatives does not change if we sum up the two types of contributions. However, by summing them, we lose the possibility to employ the RG (that is, the μ-dependence) to unveil the quantum structure of the EA.

We end up this section with a few additional comments, which are also relevant:
● If the considerations of [19] would be correct, it would mean, in particular, that we could define both induced constants CC and G in a unique way. If so, this could solve a long-standing problem of an ambiguity in the action of induced gravity [70]. Of course, the real situation is just the opposite, that is the "proof" given in [19] is empty of content and the ambiguity is there. The latter is associated to the dynamical breaking of scale invariance, hence to the trace anomaly. The induced vacuum energy reads, in the general case,

$$\rho_\Lambda^{\text{ind}}(\mu) = \frac{\Lambda^{ind}(\mu)}{8\pi \, G^{ind}(\mu)} = \frac{1}{4} \left\langle T_\alpha^\alpha(\mu) \right\rangle, \tag{60}$$

where the trace on the *r.h.s.* contains both the classical and quantum effects, which may allow for the running. Let us remember that the mentioned ambiguity means that both CC and G

depend on a single parameter which can not be defined in the framework of the initial quantum theory of matter fields. This dependence, according to [70] is universal, it has a deep physical sense and means that the induced CC and G can, in principe, run with the cosmic scale.

● We have already seen above, that the presence of a non-trivial metric can essentially change the situation, in contrast to the claims of [19]. But there is more to say. For example, in a curved space-time, the minimal interacting scalar field is not renormalizable and hence the MS-based renormalization group can not be applied [23]. If, however, we admit the presence of a non-minimal interaction of the scalar field with the curvature, the spontaneous symmetry breaking (SSB) becomes rather nontrivial and the renormalization of the theory is very complicated [39]. Moreover, the SSB generates an infinite number of non-local terms in the induced action of gravity. So, what looks somehow trivial in the simpleminded framework of [19], in reality is not so simple.

After making transparent that the "proof" of the non-running CC, as presented in [19], is insubstantial (in fact, false) in all its parts, we still have to answer the following most important question: do all the arguments presented above mean that the induced CC and G, or their vacuum counterparts, really do run with the cosmic scale? The answer is negative. Let us repeat that the renormalization group running is nothing else but a clever way to parameterize the quantum corrections. Therefore, in order to really establish the running we have to derive these corrections, or at least indicate the form which they can have. While in the MS-based RG model of [8], the non-trivial properties of the FLRW metric are encoded in the H-dependent cutoff $\mu \sim \sqrt{H\,M_P}$, one would like to have a more direct (and reliable) approach to the alleged running properties of the physical CC. In other words, one would like to have an expression for (13) in which the H-dependence is explicit, and where the decoupling effects would be manifest. A phenomenological model describing this possibility is represented e.g. by Eq. (31). However, as we have said, at the moment there is no fundamental derivation of this formula. In the next section, we address some possible roads leading to this kind of more physical relations, after studying and classifying the possible forms of the quantum effects on the EA in the context of the momentum subtraction scheme.

5   Possible forms of the quantum effects: roads to the physical running of the cosmological parameters

The purpose of this section is to discuss the possible form of the quantum corrections to the Einstein-Hilbert action and to the CC term. As we have already discussed above, such corrections should be represented by some functional of the metric and its derivatives, while other fields are in the stable vacuum states. Therefore, in this case we do not need to distinguish the vacuum and induced parts of the quantum corrections.

The classical action of vacuum for quantum matter fields is contained in the full action (1) as follows:

$$ S_{vac} = S_{EH} + S_{HD}\,, \tag{61} $$

where S_{EH} is the Einstein-Hilbert action with the cosmological constant and the second term represents a minimal set of higher derivatives necessary to insure the consistency of the theory

at the quantum level:

$$S_{HD} = \int d^4 x \sqrt{-g} \left\{ a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2 \right\} . \tag{62}$$

Here $C^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + (1/3) R^2$ is the square of the Weyl tensor and $E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\alpha\beta} + R^2$ is the integrand of the Gauss-Bonnet topological term.

Looking at the expressions (1),(61) and (62), one can ask a most natural question: how should we distribute the quantum corrections – which are typically given by non-local and non-polynomial expressions the in curvature tensor – between the different terms in the overall EA of vacuum? The first option is to perform explicit calculations of the quantum corrections, including the finite part. The history of such calculations goes back to the early works on quantum theory in curved space [71]. The most complete result in this direction has been obtained in [67] through derivation of Feynman diagrams in the framework of linearized gravity and also by using the heat kernel solution of [72]. The output of such calculation includes only the terms of zero, first and second order in the curvature tensor.

At this level, one can identify the form factors of different terms of the quantum analog of the action S_{vac} in (61) by means of analyzing their tensor structure. For example, the one-loop result for the massive real scalar field has the form [67]

$$\begin{aligned}
\bar{\Gamma}^{(1)}_{vac} &= \frac{1}{2(4\pi)^2} \int d^4 x \sqrt{-g} \left\{ \frac{m^4}{2} \cdot \left(\frac{1}{\epsilon} + \frac{3}{2} \right) + \left(\xi - \frac{1}{6} \right) m^2 R \left(\frac{1}{\epsilon} + 1 \right) \right. \\
&+ \left. \frac{1}{2} C_{\mu\nu\alpha\beta} \left[\frac{1}{60\,\epsilon} + k_W(a) \right] C^{\mu\nu\alpha\beta} + R \left[\frac{1}{2\epsilon} \left(\xi - \frac{1}{6} \right)^2 + k_R(a) \right] R \right\} .
\end{aligned} \tag{63}$$

Here ϵ is the parameter of dimensional regularization,

$$\frac{1}{\epsilon} = \frac{2}{4-n} + \ln \left(\frac{4\pi\mu^2}{m^2} \right) - \gamma_E , \tag{64}$$

and ξ is the coefficient of the non-minimal coupling. From (35) it is clear that the first term on the *r.h.s.* of (63) coincides with $-\bar{V}^{(1)}_{vac}$, see Eq. (46), i.e. the unrenormalized one-loop vacuum correction to the effective potential case considered in section 4, whose renormalized form in the $\overline{MS}$ scheme is the expression $V^{(1)}_{vac}$ given by (49). The full result (63), therefore, provides the corresponding generalization for the case when there is a non-trivial background. The presence of the curvature terms and the non-local form factors $k_W(a)$ and $k_R(a)$ – see their structure below – clearly indicates that one obtains a highly non-trivial generalization of the simple flat space-time expression studied in section 4. The essential observation is that it is no longer possible to perform an effective potential approach because some of the new terms provide non-local contributions which are explicitly dependent on the external momenta.

The mentioned form factors read, explicitly,

$$\begin{aligned}
k_W(a) &= \frac{8A}{15\,a^4} + \frac{2}{45\,a^2} + \frac{1}{150} , \\
k_R(a) &= A\left(\xi - \frac{1}{6} \right)^2 - \frac{A}{6} \left(\xi - \frac{1}{6} \right) + \frac{2A}{3a^2} \left(\xi - \frac{1}{6} \right) + \frac{A}{9a^4} - \frac{A}{18a^2} + \frac{A}{144} + \\
&+ \frac{1}{108\,a^2} - \frac{7}{2160} + \frac{1}{18} \left(\xi - \frac{1}{6} \right) ,
\end{aligned} \tag{65}$$

where we used notations

$$A = 1 - \frac{1}{a} \ln \left(\frac{2+a}{2-a} \right), \qquad a^2 = \frac{4\Box}{\Box - 4m^2} \, . \tag{66}$$

Similar expressions have been obtained for massive fermion and vector cases [67] and also for the scalar field background, where the form factors were calculated for the $\phi^2 R$ and ϕ^4 terms [76]. In the UV limit all form factors have, in general, the logarithmic behavior similar to (24), but in the IR limit they follow the Appelquist and Carazzone theorem [67, 24].

The expressions (65) suggest how the desirable quantum corrections responsible for the physical running could, in principle, look like: they should form a non-local and complicated expression, but still a relatively compact one. The form factors (65) contain a lot of information about quantum corrections in the vacuum sector, including the low-energy decoupling, conformal anomaly in the massless limit of the theory and so on [24]. However, when looking at the CC and Einstein-Hilbert sectors of (63), our optimism somehow stagnates because there is no relevant form factor in these sectors. In both cases there are divergences, there is the μ-dependence (as it should be, of course), but there is nothing real behind this μ-dependence. Indeed, we have mentioned above that the CC part is identical to the MS-based renormalization group analysis that we have presented in section 4, namely the μ-dependence of these terms is just of *Type-2*, see section 2.3. As a result, they have no correspondence with the physical renormalization group in the UV limit. In contrast, in the higher derivative sectors, we meet a perfect correspondence with the momentum subtraction scheme. Notice that, in momentum space, the form factors (65) are functions of momentum q through (66), where we have the correspondence $a^2 \to 4\,q^2/(q^2 + 4\,m^2)$.

As compared to the higher derivative terms, the origin for the unpaired result in the CC and EH case is that, when the the CC is present, the expansion around the flat background is not a perfect instrument for obtaining the quantum corrections. The form factors of the higher derivative terms do not change when we pass to the flat space background, but the ones for the CC and Einstein-Hilbert terms just vanish, because the form factors should be constructed from the d'Alembert operators, while $\Box\Lambda = 0$ and $\Box R$ is an irrelevant total derivative term. This is an explicit manifestation of the unfortunate property of the calculation based on the expansion around the flat-space. This calculational scheme can not provide information about non-local corrections to the CC and EH terms, independent of whether such corrections really exist. It is important to emphasize that there is no better technique available and, moreover, it is rather unclear how such technique may look. It is at least clear that such technique can not be related to the expansion in the powers in curvatures, because such expansion is not going to be efficient beyond the flat-space expansion framework. However, the lack of the appropriate technique by no means implies that there is no physical effect. Let us invoke, once again, the example of QCD. We know that the perturbative expansion is not efficient at the energies below, e.g. $1\,MeV$, but this does not mean that the low-energy QCD effects are irrelevant!

A good hint that the renormalization group corrections to the CC and Einstein-Hilbert terms are possible, can be obtained within the method suggested in [31] (see also [73]). The idea is to perform a conformization procedure [63] and derive the anomaly induced EA. We refer to the mentioned papers for details and just present the final result at the one-loop level

$$\bar{\Gamma} = S_c[g_{\mu\nu}] - \frac{3c + 2b}{36} \int d^4x \sqrt{-g} \, [\bar{R} - 6(\bar{\nabla}\sigma)^2 - 6(\bar{\nabla}^2\sigma)]^2$$

$$+ \int d^4x \sqrt{-\bar{g}} \, \{ w\bar{C}^2 \sigma + b \, (\bar{E} - \frac{2}{3} \bar{\nabla}^2 \bar{R}) \, \sigma + 2b \, \sigma \bar{\Delta}_4 \sigma \}$$

$$- \int d^4x \sqrt{-\bar{g}} \, e^{2\sigma} \, [\bar{R} + 6(\bar{\nabla}\sigma)^2] \cdot \left[\frac{1}{16\pi G} - f \cdot \sigma \right] - \int d^4x \sqrt{-\bar{g}} \, e^{4\sigma} \cdot \left[\frac{\Lambda}{8\pi G} - g \cdot \sigma \right], \quad (67)$$

where S_c is an unknown integration constant for the EA, and

$$\Delta_4 = \nabla^4 + 2 \, R^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{2}{3} R \nabla^2 + \frac{1}{3} (\nabla^\mu R) \nabla_\mu \tag{68}$$

is the fourth order, self-adjoint, conformal operator acting on scalars.

The action (67) is written using a special conformal parametrization of the metric $g_{\mu\nu} = e^{2\sigma(x)} \bar{g}_{\mu\nu}$, similar to the Eq. (5). Here ω, b, c, g, f are the MS-based β-functions for the parameters of the vacuum action (61). For example, f and g are dimensional quantities associated to the renormalization of the CC and EH terms:

$$f = \frac{1}{3(4\pi)^2} \sum_f N_f \, m_f^2 \,, \tag{69}$$

and

$$g = \frac{1}{2(4\pi)^2} \sum_s N_s \, m_s^4 - \frac{2}{(4\pi)^2} \sum_f N_f \, m_f^4 \,. \tag{70}$$

Here the sums are taken over all massive fermion f and scalar s fields with the masses m_f and m_s correspondingly; N_f and N_s are multiplicities of fermions and scalars.

As noticed in [31], the above expression (67) is exactly the local generalization of the renormalization group corrected classical action (61). The last is defined through the solution of the RG equation for the effective action, i.e. the solution of the PDE (19). Such solution is well-known to follow from the method of characteristics and is expressed in terms of the running charges $P(\tau)$ and fields $\Phi(\tau)$, see section 2.3. Using a notation similar to (22), the RG transformation on the renormalized EA in the MS scheme leads to

$$\Gamma[e^{-2\tau} g_{\alpha\beta}, \Phi(\mu), P(\mu), \mu, n] = \Gamma[g_{\alpha\beta}, \Phi(\tau), P(\tau), \mu, n] \,, \tag{71}$$

where, as before, $\tau = (1/2) \ln(\mu'^2/\mu^2)$. In the leading-log approximation, one can take, instead of (71), the classical action and replace (for the massless conformal theory) $P(\mu)$ with $P(\mu') = P(\mu) + \beta_P \tau$, where β_P is the β-function for the parameter P. A particular example of this expression for $P = \lambda$ is Eq. (57), with the β-function (56). Upon comparing (67) with the result of this procedure, one confirms the complete equivalence of the two expressions in the terms that do not vanish for $\sigma = const$. The important general conclusion is that the procedure [31, 73] leading to the effective action (67) is consistent with the anomaly-induced effective action and is indeed a direct generalization of the RG-improved classical action.

Let us point out that the expression (67) has been obtained in the framework of the minimal subtraction renormalization scheme, which is supposed to converge to the physical renormalization scheme at high energies. Therefore, the absence of the relevant form factors in the low-energy sectors of the expression (63) looks as an indication that the method used for deriving this expression (being equivalent to the expansion near the flat space metric) is not an

appropriate instrument for deriving the quantum correction to these terms. In contrast, the derivation of (67) does not directly rely on the expansion near the flat space metric and hence should be regarded as a kind of indirect confirmation of the existence of a nontrivial quantum corrections to CC and Einstein-Hilber terms.

In the situation when the perspective to obtain a solid quantum field theory based result is unclear, it is absolutely legitimate to use the phenomenological approach [8]-[18], investigating the role of such possible running and trying to impose some cosmological restrictions on it. However, a simple albeit very important criticism of the CC running has been presented by Dolgov [21], but in fact goes back to the epoch of L.D. Landau. If we consider that the energy-momentum tensor of the vacuum is given by (3) and (4), i.e.

$$(T_\Lambda)^\alpha_\beta = \text{diag}\left(\rho_\Lambda, \rho_\Lambda, \rho_\Lambda, \rho_\Lambda\right), \tag{72}$$

the conservation equation immediately tells us that $\rho_\Lambda = const$, as we have discussed in section 2.2. However, if we attribute a scale dependence to the CC and realize that the cosmic energy scale changes with time, we arrive at the time-dependent CC [9, 12, 15], apparently contradicting the condition $\rho_\Lambda = const$. There are nevertheless some significant loopholes in this argument. In section 2.2, we have already seen that the conservation law can be modified such that the CC is time dependent – through e.g. some relation of the type (10) – at the price of admitting non-conservation of matter or even variable gravitational coupling G. But, most important, even if there is no direct time-dependence of these quantities, the possibility of scale-dependence, as indicated in Eq. (11), can also play a crucial role.

Indeed, let us remark that, in the case of quantum corrections in curved space-time, the aforementioned conservation equation just reflects the covariance of the effective action. However, the covariance of nontrivial quantum corrections to the CC typically takes the form of nonlocal terms (see below). One can construct an infinite amount of non-local covariant actions and there is no guarantee that some of them will not give the same effect as a running CC and/or as a running G. As a particular example, we have to mention the anomaly-induced EA for the massless case, that is, the first two lines of Eq. (67), which give rise to the Starobinsky model of inflation [74] even without the CC term!

The covariance is the fundamental guiding principle in the study of vacuum quantum corrections [22, 23]. One of the important consequences of the covariance of the EA is that, when performing the expansion of this functional in the metric derivatives, there can not be terms which are odd in these derivatives [12, 24]. We have seen an example in the ansatz (31). In the cosmological setting, this implies that the quantum correction to the CC term has to be constructed from the curvature tensor, its covariant derivatives and also from the Green's functions of some covariant operators. At the low-energy cosmological scale, we can use the Hubble parameter H as the measure of this scale [9, 12, 15, 18]; in this case, the corrections can start from terms of $\mathcal{O}(H^2)$, but not from $\mathcal{O}(H)$.

The last feature suggests that the correction to the CC term in the current Universe has the form (31), where the dominant effect is $\Delta\rho_\Lambda \sim s\, M^2\, H^2$, with M^2 an average sum of the contributions of all massive particles which are present in both the observable and unobservable parts of the particle spectrum (e.g. including GUT fields with very large masses near the Planck mass); and $s = \pm 1$, depending on whether bosons or fermions dominate in the higher end of

such spectrum. This form of quantum corrections correspond to the Appelquist and Carazzone theorem [9, 10].

At this point we meet the following natural question, formulated by A. Vilenkin [20]: *In view of the fact that $R \sim H^2$, the dominant correction would be proportional to the curvature scalar, $\Delta\rho_\Lambda \sim M^2 R$, and one could naively conclude that it is not a non-trivial correction to the CC term, but rather an additive contribution to the Einstein-Hilbert term.* Again, the solution of this puzzle can be possible only because the relevant quantum effects must include nonlocalities (see below) and, thus, in general we expect that it cannot reproduce any of the local sectors of the classical action S_{vac} (61).

An example of such non-local contribution is the term in the expression (67) containing the fourth order conformal operator Δ_4. Due to the presence of the fiducial metric $\bar{g}_{\mu\nu}$, rather than the original one $g_{\mu\nu}$, the expression (67) although it looks relatively simple and compact is not manifestly covariant. Interestingly enough, when written in a covariant form (see, e.g. [55, 24]), the anomaly induced action includes the Green's functions insertions of the operator Δ_4 and hence one meets a manifestly non-local expression. As an example, the term proportional to the Weyl tensor squared, $-\bar{C}^2$, in (67) can be written in a covariant, but non-local, form as follows:

$$\frac{1}{4} \int d^4x \sqrt{-g(x)} \int d^4x' \sqrt{-g(x')} \, C^2(x) \, G(x, x') \, (E - \frac{2}{3}\Box R)_{x'} \,, \tag{73}$$

where $G(x, x')$ is the Green function of Δ_4, i.e. $\Delta_{4,x} G(x, x') = \delta(x, x')$.

The $4d$ expressions which are present in the vacuum EA, are similar to the well known Polyakov action in $2d$ [54]. However, in the present case there are very strong additional restrictions, concerning the quantum corrections to the CC and Hilbert-Einstein term. For example, these quantum contributions can not emerge in terms which are quadratic in curvatures. In order to prove this statement, let us note that the terms which do not reproduce the form of the classical vacuum action (61), should be at least $\mathcal{O}(R^2_{\ldots})$ and also have some Green's functions insertions. These insertions should correspond to the massive quantum field propagator in curved space, but since the $\mathcal{O}(R^2_{\ldots})$ terms admit the flat space expansion, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we can take the $\mathcal{O}(h_{\mu\nu})$ term for each curvature and flat-space propagator. Hence, to the order $\mathcal{O}(R^2_{\ldots})$, the result is given by the second line of Eq. (63) and its low-energy limit is characterized by the Appelquist and Carazzone - like decoupling (see detailed discussion in [67, 24]). In order to understand the phenomenological consequences, let us consider two simplified expressions which possess, qualitatively, the same property

$$R_{\mu\nu} \frac{1}{\Box + m^2} R^{\mu\nu} \,, \qquad R \frac{1}{\Box + m^2} R \,. \tag{74}$$

As far as we are always interested in the large mass limit, we can expand the propagator as

$$\frac{1}{\Box + m^2} \sim \frac{1}{m^2} \left(1 - \frac{\Box}{m^2} + \ldots\right) \,, \tag{75}$$

The first term in the parenthesis obviously gives just an additive contribution to the local structures in the vacuum action (61). Furthermore, the second term leads to the $\mathcal{O}(H^6)$ contribution and is, thus, phenomenologically irrelevant.

Let us go to higher orders in the curvature expansion. It is clear that the terms starting from $\mathcal{O}(R^3_{\ldots})$, in the low-energy cosmological setting are at least $\mathcal{O}(H^6)$ in the case where there are

only the insertions of massive Green's function. Similarly, $\mathcal{O}(R^4_{....}) \sim \mathcal{O}(H^8)$ etc. In order to obtain the relevant running of the CC and $1/G$ from the quantum corrections, a resummation in these terms must occur such that the massive Green's functions are traded for the massless ones. Therefore, we expect that the resummed result should generate a non-local contribution to the CC of the form $R\,\mathcal{F}(G_0\,R)$, for some unknown function $\mathcal{F}$ of dimension 2, where G_0 is the massless Green's function $(G_0 \sim 1/\Box)$ [77]. The canonical possibility would be $\mathcal{F} = M^2 G_0 R$, where M is the mass of the particular field involved in this term. Each one of these non-local contributions should be added to the *r.h.s.* of Eq. (13). Obviously, the heaviest fields would be dominant in that kind of non-local quantum effects, so M should mainly stand for the GUT masses of order of the Planck mass M_P. It is easy to see that this would provide an effective running CC behavior of the form $\sim M_P^2 H^2$.

It is important that such resummation, in principle, may take place in any higher order of curvature expansion or in the sum of the infinite power series of the curvature expansion. Let us note that a similar resummation does in fact take place in the anomaly-induced EA of vacuum for the massless conformal case [78]. Due to such resummation, the exact compact expression for this EA (see [75] for detailed discussion) does not involve the Green's functions of the original quantum fields (scalars, spinors and vectors) but only the Green's function of the higher derivative conformal operator (68). Also, the power of the curvature expansion of the relevant terms changes from third to fourth. Let us note that this occurrence looked impossible at the early stage of investigating the form of the anomaly-induced EA [79].

It has been known for a long time [71] that the situation for massive fields must be essentially more complicated and cumbersome compared to the massless case. It might happen that the form of the quantum contribution of our interest can not be derived in a covariant form, especially because the practical use of covariant methods implies some finite order in curvature expansion. Therefore, in view of the difficulty of practical deriving the higher order corrections, the program to prove or disprove the existence of the relevant running of the CC and $1/G$ does not look straightforward. In fact, the complicated form of the possible quantum corrections to the CC and Einstein-Hilbert terms just indicates to us (once again) that the expansion into series in curvatures is not a proper instrument for exploring the possible corrections to these terms.

Finally, from the above considerations it follows that the unique conclusion we may draw from covariance is that, if the relevant quantum corrections to the CC do exist, they have the following effective form (after imposing the boundary condition at the present time)

$$\rho_\Lambda^{ph}(H) = \rho_\Lambda^0 + \beta\,M_P^2\,(H^2 - H_0^2) + \mathcal{O}(H^4)\,, \tag{76}$$

where ρ_Λ^0 is the current value of the CC and β some numerical coefficient. This expression is precisely of the "soft-decoupling" form (31).

To summarize, the present day theoretical methods do not enable us to make a conclusive verdict about the running, except the universal form of it (76). The problem is of course very important and future work in this area should combine both things: improving our theoretical methods and also continue to work on a better understanding of the phenomenological models based on the possible CC running [8]-[18].

6   Conclusions

We have considered various quantum field theory aspects of the running of the CC term or vacuum energy density. The existing field theoretical methods can not prove that such running takes place but cannot disprove such running either. While we have no definite formal conclusion concerning the CC running, all positive statements about its existence must be done with a proper caution. However, the same is true for the negative statements, because trying to prove "non-running" theorems within an incorrect approach leads to severe errors. In particular, one such errors was committed in the recent paper [19], where a non-running statement was incorrectly supported by (extremely naive) arguments based on the μ-independence of the EA in the framework of the MS-based renormalization scheme. In reality, such μ-invariance of EA is an automatic property which holds for all kinds of renormalizable theories and, definitely, can not be used to derive any conclusions on the running.

The possibility of having a physically measurable CC running and, in general, of measuring the low-energy quantum corrections coming from massive fields is, in our opinion, one of the most important and most interesting applications of quantum field theory in the years to come. The resolution of the corresponding theoretical problems is difficult, but does not look completely inaccessible. The identification of the possible form of the vacuum effects of quantum massive fields is the key issue for the understanding of whether the variable DE in our Universe can be the consequence of the known fundamental physics or only the manifestation of a qualitatively new entities such as quintessence, k-essence, Chaplygin gas etc. Should the analysis of the astronomical data eventually give preference to a mildly variable DE, then this problem would become one of the most important ones of our time. In such case, the possibility that the cosmological parameters are running quantities would constitute the most relevant interface between quantum field theory and cosmology [9].

Acknowledgements

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